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# On indexed families of multifunctions generated by families of functions

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**Abstract.** We present some necessary and some sufficient conditions for a family of multifunctions generated by two families of real functions to be a collapsing, an expanding family or an iteration semigroup. Some properties of set-valued iteration groups generated by commuting homeomorphisms not embeddable in an iteration group are given.

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## 1. Introduction

The notion of an iteration semigroup of multifunctions was introduced and studied by Smajdor [15] and then reserched in some classes of multifunctions (see e.g. [10–13,16–18]). Afterwards Łydzińska [7] and [8] defined more general families of multifunctions the so called collapsing and expanding iteration semigroups and investigated conditions under which such families of a special form (analogous to the fundamental form of continuous iteration semigroups of single-valued functions) are iteration semigroups.

On the other hand Zdun [20], considering commuting homeomorphisms not embeddable in an iteration group, constructed an iteration group of multifunctions playing the role of a universe embedding the commuting homeomorphisms. Following the paper we repeat the construction.

Let $f, g$ be commuting continuous bijections of an open interval $I$ such that, $f(x) < x, g(x) < x$ for $x \in I$. Define $s(f, g) := \inf\{\frac{m}{n} :\ m, n \in \mathbb{N},\ f^n(x) < f^m(x)\}$, which is independent of $x \in I$ (see [6]). We assume that $s(f, g) \notin \mathbb{Q}$, consequently the set $L(f, g) := \{f^n \circ f^{-m}(x) :\ n, m \in \mathbb{N}\}$ does not depend on $x \in I$ either (cf. [19]). Moreover functions $f, g$ are iteratively incommensurable and according to Theorems 31 and 32 in [20]

$$f_-^t := \sup\{f^n \circ g^{-m} :\ n - s(f,g)\cdot m > t, n, m \in \mathbb{N}\}, \qquad t \in \mathbb{R}$$
$$f_+^t := \inf\{f^n \circ g^{-m} :\ n - s(f,g)\cdot m < t, n, m \in \mathbb{N}\}, \qquad t \in \mathbb{R},$$

are iteration groups of functions generating iteration group

$$F^t(x) = [f_-^t(x), f_+^t(x)], \qquad t \in T,\ x \in I \tag{Z}$$

of multifunctions with nonempty compact convex values.

This is a motivation to study properties of an indexed family of multifunctions $H^t : X \to cc(X)$

$$H^t(x) := [f^t(x), g^t(x)] \quad \text{for } x \in X, t \in T, \tag{H}$$

generated by families $f^t, g^t : X \to X$ of single-valued functions. We give some necessary and sufficient conditions for the family (H) to be an iteration semigroup (group) and focus on crucial properties of $f_-^t$ and $f_+^t$ implying the fact that (Z) is an iteration group. Some additional properties of the set-valued iteration group (Z) are given.

## 2. Preliminaries

Denote by $n(X)$ the family of nonempty subsets of $X \neq \emptyset, cc(X)$ the family of compact convex members of $n(X)$.

Given nonempty sets $X, Y$ and a multifunction $F : X \to 2^Y$ we define the image of a set $A \subset X$ by

$$F(A) := \bigcup_{x \in A} F(x),$$

and the lower and upper inverse images of $B \subset Y$ are respectively

$$F^-(B) := \{x \in X :\ F(x) \cap B \neq \emptyset\}, \quad F^+(B) := \{x \in X :\ F(x) \subset B\}.$$

If $X, Y$ are topological spaces, we say that $F$ is lower semicontinuous (upper semicontinuous) if $F^-(U)$ ($F^+(U)$) is open for every open set $U$. $F$ is continuous if it is both lower and upper semicontinuous (see [2,3,5]).

A single-valued function $f : X \to \mathbb{R}$ on a metric space $X$ is lower semicontinuous if $\liminf_{x \to x_0} f(x) \geq f(x_0), x_0 \in X$. The function $f$ is upper semicontinuous if $-f$ is lower semicontinuous (cf. Definition A.1.29 and Proposition A.1.30 in [5]).

We will use the abbereviations lsc and usc for lower and upper semicontinuity, respectively.

**Remark 2.1.** *According to Example 2.8 in* [5], *a multifinction $F : \mathbb{R} \to cc(\mathbb{R})$ of the form $F(x) = [f(x), g(x)], x \in \mathbb{R}$ is usc (lsc) if and only if $f$ is lsc (usc) and $g$ is usc (lsc).*

If $X \subset \mathbb{R}, Y$ is a topological space, we say that $F : X \to 2^Y$ is measurable if $F^-(U)$ is Lebesgue measurable for every open set $U$ (see [2,3,5]).

If $X, Y, Z$ are nonempty sets the superposition of multifunctions $F : X \to 2^Y$ and $G : Y \to 2^Z$ is defined as follows

$$(G \circ F)(x) = G(F(x)),\ x \in X.$$

We generalize the notion of collapsing and expanding iteration semigroups in the following way (cf. [7,8]).

**Definition 2.2.** *Let $(T, +)$ be an additive semigroup, $X \neq \emptyset$. We say that an indexed family $\{F^t : t \in T\}$ of multifunctions $F^t : X \to 2^X$ is a collapsing family if*

$$F^{s+t} \subset F^t \circ F^s \qquad \text{for } t, s \in T, \tag{C}$$

*it is an expanding family if*

$$F^t \circ F^s \subset F^{s+t} \qquad \text{for } t, s \in T. \tag{E}$$

*The family is an iteration semigroup (an extended iteration semigroup, an iteration group) if (E) and (C) are both valid for $T = (0, +\infty)$ ($T = [0, +\infty), \mathbb{R}$).*

The exponential family of linear continuous multifunctions investigated in [14] is an expanding family. Necessary and sufficient conditions for the family to be an iteration semigroup are given in [13].

We say that a family $\{F^t :\ t \in T\}$ ($T = (0, +\infty), [0, +\infty), \mathbb{R}$) is usc (lsc, measurable) if the multifunction $t \mapsto F^t(x)$ is usc (lsc, measurable) for every $x \in X$.

**Lemma 2.3.** *Let $t_0 \in \mathbb{R}$ and let $\{F^t : t \in \mathbb{R}\}$ be an expanding family of multifunctions $F^t : X \to n(X)$. If $F^{t_0}(x) = \{x\}$ for $x \in X$ then the family $\{F^t : t \in \mathbb{R}\}$ is a single-valued iteration group.*

*Proof.* Fix any $t \in \mathbb{R}$ and $x \in X$. Since $F^{t_0-t}(F^t(x)) \subset F^{t_0}(x) = \{x\}$ and the values of $F^t$ and $F^{t_0-t}$ are nonempty

$$F^{t_0-t}(F^t(x)) = \{x\}.$$

Therefore $F^{t_0-t}(y) = \{x\}$ for every $y \in F^t(x)$. Moreover, taking any $y \in F^t(x) \neq \emptyset$ we obtain

$$\{y\} \subset F^t(x) = F^t(F^{t_0-t}(y)) \subset F^{t_0}(y) = \{y\}.$$

Consequently the values of every $F^t$ are singletons and $F^t \circ F^s = F^{s+t}$ for $t, s \in\ \mathbb{R}$. $\square$

For a family $\{F^t : t \in \mathbb{R}\}$ of multifunctions $F^t : X \to 2^X$ define the multifunction $c_F : X \times X \to 2^T$ (cf. [15, Definition 3.1] for iteration semigroups) as follows

$$c_F(x, y) := \{t \in T :\ y \in F^t(x)\},\quad x, y \in X, \tag{2.1}$$

and denote conditions

(a) for every $t \in T$ and $x \in X$ there exists $y \in X$ such that $t \in c_F(x, y)$,

(b) $c_F(x, y) + c_F(y, z) \subset c_F(x, z)$, for $x, y, z \in X$,

(c) if $x, z \in X$, $t, s \in T$ and $s + t \in c_F(x, z)$,
then there exists $y \in X$ such that $s \in c_F(x, y)$ and $t \in c_F(y, z)$.

Then analogously to the proof of Theorem 3.1 in [15] we get properties of the multifunction $c_F$.

**Lemma 2.4.** *Let $X$ be a nonempty set, $\{F^t : t \in T\}$ be a family of multifunctions $F^t : X \to 2^X$. Then*

$$\begin{aligned}
&(\mathrm{a}) \iff F^t \textit{ has nonempty values for every } t \in T,\\
&(\mathrm{b}) \iff \{F^t : t \in T\} \textit{ is an expanding family},\\
&(\mathrm{c}) \iff \{F^t : t \in T\} \textit{ is a collapsing family.}
\end{aligned}$$

*Proof.* The first statement follows immediately from the definition of $c_F$.

Now assume (b) holds and take $t, s \in T, x \in X$. If $F^t \circ F^s(x) = \emptyset$ then $F^t \circ F^s(x) \subset F^{s+t}(x)$. If $z \in F^t \circ F^s(x)$ then there exists $y \in F^s(x)$ such that $z \in F^t(y)$ which means $s \in c_F(x, y)$ and $t \in c_F(y, z)$. Thus by (b)

$$s + t \in c_F(x, y) + c_F(y, z) \subset c_F(x, z),$$

which yields $z \in F^{s+t}(x)$.

On the other hand, suppose that $\{F^t : t \in T\}$ is an expanding family and fix $x, y, z \in X$. If one of the sets $c_F(x, y), c_F(y, z)$ is empty, then (b) is fulfilled. In case $s \in c_F(x, y), t \in c_F(y, z)$ we have

$$z \in F^t(y) \subset F^t(F^s(x)) \subset F^{s+t}(x)$$

and consequently $s + t \in c_F(x, z)$.

Assuming that (c) holds, take $s, t \in T$ and $x \in X$. Obviously, in case $F^{s+t}(x) = \emptyset$ we have $F^{s+t}(x) \subset F^t \circ F^s(x)$. If there exists $z \in F^{s+t}(x)$, then $s + t \in c_F(x, z)$ and on account of (c)

$$s \in c_F(x, y), t \in c_F(y, z)$$

for some $y \in X$. Therefore $z \in F^t(y) \subset F^t(F^s(x))$.

Finally assume that $\{F^t : t \in T\}$ is a collapsing family. Then for fixed $x, z \in X, s, t \in T$ such that $s + t \in c_F(x, z)$

$$z \in F^{s+t}(x) \subset F^t \circ F^s(x).$$

Consequently, there exists $y \in X$ such that $z \in F^t(y)$ and $y \in F^s(x)$, hence

$$t \in c_F(y, z), s \in c_F(x, y),$$

which completes the proof. $\square$

For the convenience of the reader we recall another property of $c_F$ which will be needed in the last section.

**Lemma 2.5 (Theorem 3.8, [15]).** *Let $X$ be a nonempty set, $\{F^t : t \in T\}$ be an iteration semigroup (an extended iteration semigroup) of multifunctions $F^t : X \to n(X)$ such that sets $c_F(x, y), x, y \in X$ are closed intervals. Then the functions $F^t$ are of the form*

$$F^t(x) = \{y \in X : \inf c_F(x, y) \leq t \leq \sup c_F(x, y)\}.$$

If $X$ is a linear space we denote by $\operatorname{conv} A$ the convex hull of $A \subset X$, which is the smallest (in the sense of inclusion) convex set containing $A$.

Let $I = [a, b], J = [c, d] \in cc(\mathbb{R})$. We say that

$$I \preceq J :\Longleftrightarrow a \leq c \text{ and } b \leq d.$$

**Definition 2.6.** *Let $X \subset \mathbb{R}$. We say that a multifunction $F : X \to cc(\mathbb{R})$ is nondecreasing if*

$$F(x) \preceq F(y) \quad \text{for } x, y \in X \text{ such that } x \leq y.$$

## 3. Main results

Now assume that $(T, +)$ is an additive semigroup, $X \subset \mathbb{R}$ and $f^t, g^t : X \to X$. We investigate relationships between properties of the family (H) and the following conditions:

$$f^t \leq g^t \quad \text{for every } t \in T, \tag{H1}$$

$$f^t, g^t \quad \text{are nondecreasing for every } t \in T, \tag{N}$$

$$f^t \circ f^s \geq f^{s+t},\ g^t \circ g^s \leq g^{s+t} \quad \text{for every } t, s \in T, \tag{E1}$$

$$f^t \circ f^s \leq f^{s+t},\ g^t \circ g^s \geq g^{s+t} \quad \text{for every } t, s \in T. \tag{C1}$$

Families of functions satisfying (H1) generate a family of the form (H) and vice versa a family $\{H^t : t \in T\}$ of multifunctions $H^t : X \to cc(X)$ generates two families of single-valued functions $\{f^t : t \in T\}, \{g^t : t \in T\}$ satisfying (H1).

**Remark 3.1.** *The family (H) consists of nondecreasing multifunctions iff (N) is valid.*

**Proposition 3.2.** *Functions generating an expanding family (H) fulfill (E1).*

*Proof.* Observe that $f^t(x), g^t(x) \in H^t(x)$ for every $x \in X$ and $t \in T$. Therefore for fixed $x \in X$ and $t, s \in T$

$$f^t(f^s(x)) \in H^t(f^s(x)) \subset H^t(H^s(x)) \subset H^{s+t}(x) = [f^{s+t}(x), g^{s+t}(x)],$$

which implies $(f^t \circ f^s)(x) \geq f^{s+t}(x)$. Analogously

$$g^t(g^s(x)) \in H^t(g^s(x)) \subset H^t(H^s(x)) \subset H^{s+t}(x) = [f^{s+t}(x), g^{s+t}(x)]$$

and finally $(g^t \circ g^s)(x) \le g^{s+t}(x)$. □

**Proposition 3.3.** *Functions generating a collapsing family* (H) *of nondecreasing multifunctions fulfill* (C1)*.*

*Proof.* Let $x \in X, t, s \in T$. Since

$$f^{s+t}(x), g^{s+t}(x) \in H^{s+t}(x) \subset H^t(H^s(x)),$$

there exist $y, z \in H^s(x) = [f^s(x), g^s(x)]$ such that

$$f^{s+t}(x) \in H^t(y) = [f^t(y), g^t(y)] \text{ and } g^{s+t}(x) \in H^t(z) = [f^t(z), g^t(z)].$$

Therefore by the monotonicity of $f^t$ and $g^t$

$$f^t(f^s(x)) \le f^t(y) \le f^{s+t}(x), \;\; g^{s+t}(x) \le g^t(z) \le g^t(g^s(x)).$$

□

According to Remark 3.1, a collapsing family (H) of nondecreasing multifunctions fulfills (N) which is essential for the validity of the above proposition, as is shown in the following examples.

**Example 1.** *Let* $T = \mathbb{R}, I = [0,1]$, (H) *be generated by functions* $f^t(x) = \begin{cases} \frac{1}{2} & \text{for } x \in [0, \frac{1}{2}] \\ 0 & \text{for } x \in (\frac{1}{2}, 1] \end{cases}, g^t(x) = \begin{cases} \frac{1}{2} & \text{for } x \in [0, \frac{1}{2}] \\ 1 & \text{for } x \in (\frac{1}{2}, 1] \end{cases}, t \in T$. *Since* $f^t$ *are not nondecreasing, the condition* (N) *is not satisfied and neither is* (C1)*, although* (H) *is a collapsing family.*

**Example 2.** *Let* $T = [1, +\infty), I = [0, +\infty)$, (H) *be generated by functions* $f^t \equiv 0, g^t(x) = \begin{cases} 1 - \frac{1}{t}x & \text{for } x \le t \\ 0 & \text{for } x > t \end{cases}, t \in T$. *Since* $g^t$ *are not nondecreasing, the condition* (N) *is not satisfied. Observe that* $H^s(0) = [0,1]$ *and* $0 \in H^s(x)$ *for every* $x \in [0, +\infty), s \in T$*, therefore*

$$H^t(H^s(x)) = [0,1] \supset H^{s+t}(x) \quad \text{for } t \in T$$

*and consequently* (H) *is a collapsing family. On the other hand* (C1) *is not valid, since*

$$g^t\left(g^1\left(\frac{1}{2}\right)\right) = 1 - \frac{1}{2t} < 1 - \frac{1}{2(t+1)} = g^{t+1}\left(\frac{1}{2}\right).$$

**Corollary 3.4.** *If a family* (H) *of nondecreasing multifunctions fulfills*

$$H^{s+t} = H^t \circ H^s \quad \text{for } t, s \in T \tag{3.1}$$

*then* (H1)*–*(C1) *are satisfied.*

**Proposition 3.5.** *If* (H1)*–*(E1) *are valid, then* (H) *is an expanding family.*

*Proof.* Take any $x \in X, t, s \in T$ and $y \in H^t \circ H^s(x)$. Then there exists $z \in H^s(x)$ such that

$$y \in H^t(z) = [f^t(z), g^t(z)]. \tag{3.2}$$

Since $f^s(x) \le z \le g^s(x)$, by our assumptions we have

$$f^{s+t}(x) \le f^t(f^s(x)) \le f^t(z) \le g^t(z) \le g^t(g^s(x)) \le g^{s+t}(x),$$

which with (3.2) yields $y \in H^{s+t}(x)$ and completes the proof. □

The following examples show that the assumption on the monotonicity of functions generating (H) is essential.

**Example 3.** *Let $T = \mathbb{R}, I = [0,1]$, (H) be generated by functions $f^t \equiv 0, g^t(x) = \begin{cases} 1-x & \text{for } x \in [0,\frac{1}{2}] \\ x & \text{for } x \in (\frac{1}{2},1] \end{cases}, t \in T$ satisfying only (H1) and (E1) ($g^t$ are not nondecreasing ). Since $0 \in H^s(\frac{1}{2})$ for $s \in T$ and*

$$H^{t+s}\left(\frac{1}{2}\right) = \left[0, \frac{1}{2}\right] \subset [0,1] = H^t(0) \subset \bigcup_{y \in H^s(\frac{1}{2})} H^t(y) = H^t \circ H^s\left(\frac{1}{2}\right), \quad t \in T$$

*(H) is not expanding.*

**Example 4.** *Let $T = [1, +\infty), I = [0,1]$, (H) be generated by functions $f^t(x) = 0, x \in [0,1], g^t(x) = \begin{cases} 1 & \text{for } x \in [0,\frac{1}{2}] \\ 1-\frac{1}{t} & \text{for } x \in (\frac{1}{2},1] \end{cases}, t \in T$ satisfying (H1) and (E1). Observe that $g^t$ are not nondecreasing (the condition (N) is not fulfilled). On the other hand $0 \in H^s(x)$ for every $x \in [0,1]$ and $s \in T$, therefore $H^t(H^s(x)) = [0,1]$ for $t \in T$. Consequently (H) is not expanding, since $H^t \circ H^s(x)$ is not contained in $H^{s+t}(x) = [0, 1 - \frac{1}{s+t}]$ for all $x > \frac{1}{2}$.*

Combining Lemma 2.3 and Proposition 3.5 we get the following corollary.

**Corollary 3.6.** *If (H1)–(E1) are valid with $T = \mathbb{R}$ and additionaly $f^{t_0} = g^{t_0} = id$ for some $t_0 \in \mathbb{R}$, then the family (H) is a single-valued iteration group.*

The family (H) generated by nondecreasing functions fulfilling (C1) need not be collapsing, as is shown in the following example.

**Example 5.** *Let $T = [2, +\infty), I = [0,1]$, let (H) be generated by nondecreasing functions $f^t(x) = \begin{cases} 0 & \text{for } x \in [0,\frac{1}{2}] \\ \frac{1}{t} & \text{gdy } x \in (\frac{1}{2},1] \end{cases}, g^t(x) = \begin{cases} 0 & \text{for } x \in [0,\frac{1}{2}] \\ 1 & \text{for } x \in (\frac{1}{2},1] \end{cases}, t \in T$ satisfying (H1) and (C1). Then (H) is not collapsing, since*

$$H^t(H^s(1)) = \{0\} \cup \left[\frac{1}{t}, 1\right] \quad \text{and} \quad H^{s+t}(1) = \left[\frac{1}{s+t}, 1\right] \quad \text{for all } t, s \ge 2.$$

**Proposition 3.7.** *If (H) is a family generated by functions satisfying (C1) and every $H^t, t \in T$ is lsc or usc, then it is a collapsing family.*

*Proof.* Take any $x \in X, t, s \in T$. Observe that by (H1) and (C1)

$$f^t \circ f^s \leq f^{s+t} \leq g^{s+t} \leq g^t \circ g^s.$$

Moreover

$$f^t(f^s(x)) \in H^t(f^s(x)) \subset H^t \circ H^s(x), \ \ g^t(g^s(x)) \in H^t(g^s(x)) \subset H^t \circ H^s(x)$$

and therefore

$$\begin{aligned} H^{s+t}(x) &= [f^{s+t}(x), g^{s+t}(x)] \subset [f^t(f^s(x)), g^t(g^s(x))] \\ &= \operatorname{conv}\{f^t(f^s(x)), g^t(g^s(x))\} \subset \operatorname{conv} H^t \circ H^s(x). \end{aligned}$$

Since $H^t$ is usc or lsc and $H^s(x)$ is connected, according to Proposition 2.24 in [5], $H^t \circ H^s(x) = H^t(H^s(x))$ is connected and consequently convex, which completes the proof. □

**Example 6.** *Functions $f^t(x) = 0, g^t(x) = x^2, x \in [0,1], t \in T$ fulfill (H1)–(E1) and by Proposition 3.5 the family (H) of nondecreasing multifunctions is expanding. Since (C1) is not satisfied, according to Proposition 3.3 it is not collapsing.*

The family defined in the above example is not of the form (A) considered in [9, Theorem].

**Example 7.** *The family (H) generated by functions $f^t(x) = 0, g^t(x) = \sqrt{x}, x \in [0,1], t \in T$ fulfills the assumptions of Proposition 3.7 and therefore is collapsing. On account of Proposition 3.2 the family is not expanding, since (E1) is not fulfilled.*

Propositions 3.5 and 3.7 yield the following corollary.

**Corollary 3.8.** *If (H) is a family generated by functions satisfying (H1)–(C1) and every $H^t, t \in T$ is lsc or usc, then*

$$H^t \circ H^s = H^{s+t} \quad \textit{for } t, s \in T.$$

*Moreover, if $T = \mathbb{R}$ the family (H) is an iteration group of multifunctions, if $T = (0, +\infty)$ it is an iteration semigroup and in the case $T = [0, +\infty)$ this family is an extended iteration semigroup.*

## 4. Applications to a multivalued iteration group generated by commuting functions

Applying our results, we have another proof of the fact that the family (Z) is an iteration group of multifunctions. Indeed, observe that functions $f_-^t, f_+^t$ are respectively lsc and usc for $t \in \mathbb{R}$ (cf. [20, proof of Lemma 7]) and whence every $F^t = [f_-^t, f_+^t]$ is usc (see Remark 2.1). By Lemma 7, Remark 9 and Lemma 16 in [20], $f_-^t, f_+^t, t \in \mathbb{R}$ satisfy the assumptions of Corollary 3.8 with $T = \mathbb{R}$.

Moreover, according to the properties of functions $t \mapsto f_-^t(x), t \mapsto f_+^t(x), x \in I$ and the following lemma we obtain the measurability of the iteration group (Z), which is particular interesting in case $\operatorname{Int} L(f,g) = \emptyset$ (see [19,20]).

**Lemma 4.1.** *Let $I \subset \mathbb{R}$ be an open interval. If $f : I \to \mathbb{R}$ is nonincreasing and right(left)-hand side continuous then it is lsc (usc).*

*Proof.* Assume that $f$ is nonincreasing and right-hand side continuous and fix $x_0 \in I$. By the monotonicity of $f, f(x) \geq f(x_0)$ for every $x < x_0$. Therefore $\liminf_{x \to x_0^-} f(x) \geq f(x_0)$. According to the right-hand side continuity of $f$, we have $\lim_{x \to x_0^+} f(x) = f(x_0)$. It follows that $\liminf_{x \to x_0} f(x) \geq f(x_0)$, hence $f$ is lsc. The proof is similar when $f$ is left-hand side continuous. □

**Remark 4.2.** *The iteration group* (Z) *is usc and therefore measurable.*

*Proof.* Let $x \in I$. On account of Lemma 11 and Lemma 28 in [20], the function $t \mapsto f_-^t(x)$ is strictly decreasing and right-hand side continuous and therefore lsc, the function $t \mapsto f_+^t(x)$ is strictly decreasing and left-hand side continuous, consequently usc on I (see Lemma 4.1). According to Remark 2.1 the multifunction $t \mapsto F^t(x) = [f_-^t(x), f_+^t(x)]$ is usc, finally by Proposition 8.2.1 in [2] it is measurable. □

**Theorem 4.3.** *Let $\{F^t : t \in \mathbb{R}\}$ be the iteration group given by* (Z)*, $c_F$ be the multifunction defined by* (2.1)*. Then*

(a) *$c_F$ is a single-valued function,*

(b) *$c_F(x,z) = c_F(x,y) + c_F(y,z)$, for $x, y, z \in I$,*

(c) *if $x, z \in I$, $t, s \in \mathbb{R}$ and $s + t = c_F(x,z)$,*
*then there exists $y \in I$ such that $s = c_F(x,y)$ and $t = c_F(y,z)$,*

(d) *$F^t(x) = \{y \in I : t = c_F(x,y)\}$.*

*Proof.* Fix $x, y \in I$. Combining the conditions (A2) and (A7) of Theorem 38 in [20] there exists exactly one $t \in \mathbb{R}$ such that $y \in F^t(x)$, thus $c_F(x,y)$ is a singleton.

Lemma 2.4 with the fact that $c_F$ is single-valued yield (b) and (c).

Note that $\{F^t : t > 0\}$ and $\{F^{-t} : t \geq 0\}$ are iteration semigroups (the second one is extended). Since $c_F(x,y)$ are closed intervals (as singletons), the condition (d) is a consequence of Lemma 2.5 which completes the proof. □

The condition (b) means that $c_F$ is a solution of Sincov's functional equation and therefore is of the form $c_F(x,y) = h(y) - h(x)$, where $h : I \to \mathbb{R}$ is an arbitrary function (see [1,4]). In this particular case we have more information on $h$. Consider the system of Abel equations

$$\begin{cases} \varphi(f(x)) = \varphi(x) + 1 \\ \varphi(g(x)) = \varphi(x) + s(f,g) \end{cases}, \quad x \in I. \tag{4.1}$$

According to Proposition 6 in [20], the system (4.1) has a unique continuous solution (up to an additive constant). The solution is nonincreasing (it is invertible iff $\operatorname{Int} L(f,g) \neq \emptyset$).

**Theorem 4.4.** *Let $\{F^t : t \in \mathbb{R}\}$ be the iteration group given by (Z), $c_F$ be defined by (2.1), $\varphi$ be a continuous solution of the system (4.1). Then*

(i) $c_F(x,y) = \varphi(y) - \varphi(x)$, *for* $x, y \in I$,
(ii) $c_F$ *is continuous,*
(iii) $c_F(\cdot, y)$ *is nondecreasing,* $c_F(x, \cdot)$ *is nonincreasing for* $x, y \in I$,
(iv) $F^t(x) = \{y \in I : \varphi(y) = \varphi(x) + t\}$ *for* $x \in I$, $t \in \mathbb{R}$
(v) $F^t(x) = \varphi^{-1}(\{\varphi(x) + t\})$ *for* $x \in I$, $t \in \mathbb{R}$.

*Proof.* On account of Remark 33 in [20], $\varphi$ also satisfies the system of equations

$$\varphi(F^t(x)) = \{\varphi(x) + t\}, \ \ t \in \mathbb{R}, \ x \in I.$$

Therefore taking any $x, y \in I$ and $t = c_F(x,y)$ we have $y \in F^t(x)$ and thus

$$\varphi(y) = \varphi(x) + t = \varphi(x) + c_F(x,y)$$

which yields (i) and (iv).

From (i) and the above mentioned properties of $\varphi$ we have (ii) and (iii).

It remains to prove (v). Obviously, $F^t(x) \subset \varphi^{-1}(\{\varphi(x) + t\})$. On the other hand, if $y \in \varphi^{-1}(\{\varphi(x)+t\})$, then $\varphi(y) = \varphi(x) + t$. By (i)

$$t = \varphi(y) - \varphi(x) = c_F(x,y)$$

which implies that $y \in F^t(x)$ and the proof is complete. □

Note that the iteration group (Z) is of the form (v) studied in [8].



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